

# Toroidal AdS Charged Branes and Toda Equations

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## Abstract

In this note, we consider the equations of motion for charged branes in AdS space and show that they can be cast into one-dimensional coupled Toda type. Then we solve the equations of motion to construct static charged AdS brane solutions which are invariant under translation along  $p$  directions and have toroidal symmetry. The solutions are described by mass, charges and a dilaton coupling constant.

Keywords : black brane; Toda equation.

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# 1 Introduction

The holographic principle is a speculative conjecture about quantum gravity claiming that all the information contained in a volume of space can be represented by information living in the boundary of that region [1, 2]. The principle was formulated as an attempt to understand the physics of quantum black holes and to reconcile gravitational collapse and unitarity of quantum mechanics at the Planck scale. Thus, it is very tempting to consider the principle as one of organizing principles for quantum gravity. String theory provides the most concrete realization of the holographic principle for spacetimes with negative cosmological constant, namely the AdS/CFT correspondence, which is a nonperturbative background-independent definition of quantum gravity in asymptotically AdS space [3, 4, 5]. The AdS/CFT correspondence is a conjecture about the duality of classical supergravity and string theory on the  $\text{AdS}_{d+1}$  times a compact manifold and certain conformal field theories in  $d$  dimensions in the large  $N$  limit. Thus AdS/CFT correspondence makes AdS black holes still attractive.

Meanwhile there is only a very limited family of asymptotically flat, stationary black hole solutions to Einstein equations in four dimensions, in higher dimensions there exist different kinds of black objects such as black strings, black branes, Kaluza-Klein black holes, Kaluza-Klein bubbles and black tubes as well as conventional black holes with hyperspherical horizons,  $S^n$ . Black branes in more than four dimensions are of particular interest, since they exhibit new behaviors that black holes do not show. The black branes are solutions which are extended in extra  $p$  spatial dimensions and do not diverge at spatial infinity [6, 7, 8, 9, 10, 11, 12, 13]. Many brane solutions have been constructed for simple truncations of supergravity theories. Similarly to the black hole solution in four dimensions, a black brane may carry electric and/or magnetic charges and couple to a dilaton field.

Usually the black branes have complicated metrics, and it is difficult to solve the second-order differential equations of motion. However, in this paper, we will apply the effective action method and express two equations of motion in terms of one-dimensional coupled Toda equations. Integration of the system leads to a generic solution containing several integration constants. In the literature it was claimed that extra parameters other than charges and the event horizon radius may be associated with additional physical structures. However, the detailed investigation of the geometric structure of the solution in many cases revealed that the additional parameters lead to

naked singularities [14, 15, 16]. Then our solutions can be shown to be characterized by mass density, magnetic charges and a coupling constant after suitable coordinate transformation.

The structure of the paper is as follows. In section 2 we will obtain the a AdS magnetically charged brane solution with toroidal symmetry  $T^n$  by solving equations of motion. The procedure to derive an electrically charged brane solution in AdS space would be very similar. However, in this paper, we will consider only magnetically charged solution. Then, in section 3, the result is extended to a dilatonic deformation. The section 4 is reserved for the discussions.

## 2 Charged Brane in AdS Space

In this section we will obtain static, toroidal, magnetically charged brane solutions in AdS space. We start with the  $(n + p + 2)$ -dimensional AdS-Einstein-Maxwell action,

$$S = \int \frac{d^{n+p+2}x \sqrt{-g}}{16\pi G_{n+p+2}} \left[ \mathcal{R} - \frac{1}{2} F^{M\nu_1 \dots \nu_n} F^M_{\nu_1 \dots \nu_n} + \frac{(n+p)(n+p+1)}{l^2} \right], \quad (1)$$

where  $M$  runs over  $M = 1, \dots, N_m$  and  $N_m$  is the number of magnetic fields. Now we take the following metric ansatz for a static solution which has translation symmetry along  $p$  directions and toroidal symmetry,

$$ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + e^{2C(r)} \sum_{i=1}^n d\theta_i^2 + e^{2F(r)} \sum_{i=1}^p dz_i^2, \quad (2)$$

where all the metric fields are functions of  $r$  only. We need to solve the Maxwell equations first, and then plug the result for the gauge fields into the action. The field strength ansatz consistent with the symmetries is

$$F^M_{\theta_1 \dots \theta_n} = P^M, \quad (3)$$

by which the Maxwell equations are automatically satisfied. Then, arranging the terms in squared forms we get the following effective Lagrangian density,

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & e^{A(r)-B(r)+pF(r)+nC(r)} \left[ \{A'(r) + pF'(r) + nC'(r)\}^2 + \frac{(n+p)(n+p+1)}{l^2} e^{2B(r)} \right. \\ & - \frac{1}{p+1} \{A'(r) + pF'(r)\}^2 - \frac{q^2}{2} e^{2\{B(r)-nC(r)\}} - \frac{p}{p+1} \{A'(r) - F'(r)\}^2 \\ & \left. - n\{C'(r)\}^2 \right] - 2 \left[ e^{A(r)-B(r)+pF(r)+nC(r)} \left\{ A'(r) + pF'(r) + nC'(r) \right\} \right]' . \end{aligned} \quad (4)$$

where  $q = \sqrt{\sum_{M=1}^{N_m} (P^M)^2}$ .

Since there is no  $B'(r)$  term in the effective Lagrangian (4),  $B(r)$  is not a dynamical variable, but merely gives us a constraint condition,

$$\begin{aligned} & \{A'(r) + pF'(r) + nC'(r)\}^2 - \frac{(n+p)(n+p+1)}{l^2} e^{2B(r)} - \frac{p}{p+1} \{A'(r) - F'(r)\}^2 \\ & - \frac{1}{p+1} \{A'(r) + pF'(r)\}^2 + \frac{q^2}{2} e^{2\{B(r)-nC(r)\}} - n\{C'(r)\}^2 = 0. \end{aligned} \quad (5)$$

Then taking the gauge  $B(r) = A(r) + pF(r) + nC(r)$  after varying the effective Lagrangian (4) with respect to the other fields, the equations of motion can be arranged as follows,

$$\begin{aligned} & \{A''(r) + pF''(r)\} + (n-1)\{A''(r) + pF''(r) + nC''(r)\} \\ & = \frac{n(n+p)(n+p+1)}{l^2} e^{2\{A(r)+pF(r)+nC(r)\}}, \end{aligned} \quad (6)$$

$$\begin{aligned} & (n+p+1)\{A''(r) + pF''(r)\} - (p+1)\{A''(r) + pF''(r) + nC''(r)\} \\ & = \frac{n(p+1)q^2}{2} e^{2\{A(r)+pF(r)\}}, \end{aligned} \quad (7)$$

$$A''(r) - F''(r) = 0. \quad (8)$$

The first two equations (6), (7) are one-dimensional coupled Toda equations, which give us four integration constants. The last equation (8) can be readily solved, which gives two integration constants. Then totally we have six integration constants, and four among them can be set by utilizing the definition of  $r$  coordinate, overall constant rescaling of the metric and the symmetries of time and  $z_i$  coordinate rescalings. Because the constraint (5) gives us a relation, a combination of parameters of the general solution of the equations of motion should satisfy it. Thus, we are left with a  $(2 + N_m)$ -parameter family of solutions [17]. More generally, unless we assume the uniformity, a solution with  $n+p$  nonuniform tensions has  $n+p+N_m$  parameters. However, the regularity conditions narrow meaningful solutions by demanding definite tension parameters [18, 19].

Now we want to find an exact solution of the equations of motion (6), (7), (8). The last equation (8) can be easily solved,

$$A'(r) - F'(r) + \alpha = 0 \Rightarrow A(r) - F(r) = -\alpha r + \beta, \quad (9)$$

where  $\alpha$  and  $\beta$  are integration constants. The first and the second equations (6), (7) can be arranged as follows,

$$J''(r) = e^{\frac{2(n+p+1)}{n+p}J(r) - \frac{2}{n+p}K(r)}, \quad (10)$$

$$K''(r) = e^{\frac{2(p+1)}{n+p}K(r) + \frac{2(n-1)}{n+p}J(r)}, \quad (11)$$

where

$$J(r) = \frac{1}{n}\{A(r) + pF(r)\} + \frac{n-1}{n}\{A(r) + pF(r) + nC(r)\} + \dots, \quad (12)$$

$$K(r) = \frac{n+p+1}{n}\{A(r) + pF(r)\} - \frac{p+1}{n}\{A(r) + pF(r) + nC(r)\} + \dots \quad (13)$$

and  $\dots$  represents some suitable constants to make eq.(10) and eq.(11). There is a special solution with  $J(r) = K(r)$ ,

$$e^{-J(r)} = e^{-K(r)} = \frac{1}{\gamma} \sinh(\gamma r + \delta) \quad (14)$$

where  $\gamma$  and  $\delta$  are integration constants. By setting  $\alpha = \gamma$ ,  $\beta = \delta = 0$  and transforming the  $r$  coordinate to  $\rho$ ,  $r = \frac{1}{2\alpha} \ln \frac{\rho^2}{\rho^2 - r_+^2}$ , it is found that this solution stands for the product of a  $(p+2)$ -dimensional black brane and an  $n$ -dimensional torus. In general, there would exist a full  $(n+p+2)$ -dimensional black brane solution which interpolates between this special one near the horizon and AdS space at spatial infinity.

### 3 Dilatonic Charged Brane in AdS Space

We can extend the result in section 2 to the case of a brane solution which couples to a neutral dilatonic scalar field. Let us take into account the following  $(n+p+2)$ -dimensional AdS-Einstein-Maxwell-dilaton action,

$$S = \int \frac{d^{n+p+2}x \sqrt{-g}}{16\pi G_{n+p+2}} \left[ \mathcal{R} - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} e^{a\phi} F^{M\nu_1 \dots \nu_n} F^M_{\nu_1 \dots \nu_n} + \frac{(n+p)(n+p+1)}{l^2} \right], \quad (15)$$

where  $a$  characterizes the strength of the dilaton. Now we take the same metric ansatz (2), and the same field strength ansatz (3) consistent with the

symmetries, which satisfies the Maxwell equations automatically. Plugging these ansatzes into the above action, we find the effective action,

$$\begin{aligned}
\mathcal{L}_{\text{eff}} = & e^{A(r)-B(r)+pF(r)+nC(r)} \left[ \{A'(r) + pF'(r) + nC'(r)\}^2 + \frac{(n+p)(n+p+1)}{l^2} e^{2B(r)} \right. \\
& - \frac{2}{2(p+1)+a^2} \{A'(r) + pF'(r) + \frac{a}{2}\phi'(r)\}^2 - \frac{q^2}{2} e^{a\phi(r)+2\{B(r)-nC(r)\}} \\
& - \frac{p}{p+1} \{A'(r) - F'(r)\}^2 - \frac{a^2}{(p+1)\{2(p+1)+a^2\}} \{A'(r) + pF'(r) - \frac{p+1}{a}\phi'(r)\}^2 \\
& \left. - n\{C'(r)\}^2 \right] - 2 \left[ e^{A(r)-B(r)+pF(r)+nC(r)} \left\{ A'(r) + pF'(r) + nC'(r) \right\} \right]'.
\end{aligned} \quad (16)$$

If we considered electric fields instead of magnetic ones, the sign of the coupling constant  $a$  or the scalar field  $\phi$  would be opposite in the above expression. So, the procedure to derive a static, dilatonic, electrically charged brane solution in AdS space would be very similar to the following. Since the above effective Lagrangian (16) does not have any  $B'(r)$  term, the variation with respect to  $B(r)$  only gives us a constraint condition,

$$\begin{aligned}
& \{A'(r) + pF'(r) + nC'(r)\}^2 - \frac{(n+p)(n+p+1)}{l^2} e^{2B(r)} - \frac{p}{p+1} \{A'(r) - F'(r)\}^2 \\
& - \frac{2}{2(p+1)+a^2} \{A'(r) + pF'(r) + \frac{a}{2}\phi'(r)\}^2 + \frac{q^2}{2} e^{a\phi(r)+2\{B(r)-nC(r)\}} \\
& - n\{C'(r)\}^2 - \frac{a^2}{(p+1)\{2(p+1)+a^2\}} \{A'(r) + pF'(r) - \frac{p+1}{a}\phi'(r)\}^2 = 0.
\end{aligned} \quad (17)$$

First varying the effective Lagrangian (16) with respect to each field and then choosing the gauge  $B(r) = A(r) + pF(r) + nC(r)$ , we have the following equations of motion,

$$\begin{aligned}
& \{A''(r) + pF''(r) + \frac{a}{2}\phi''(r)\} + \frac{2(n-1)(p+1) + (n+p)a^2}{2(p+1)} \{A''(r) + pF''(r) + nC''(r)\} \\
& = \frac{(n+p)(n+p+1)\{2n(p+1) + (n+p+1)a^2\}}{2(p+1)l^2} e^{2\{A(r)+pF(r)+nC(r)\}},
\end{aligned} \quad (18)$$

$$\begin{aligned}
& (n+p+1)\{A''(r) + pF''(r) + \frac{a}{2}\phi''(r)\} - (p+1)\{A''(r) + pF''(r) + nC''(r)\} \\
& = \frac{q^2\{2n(p+1) + (n+p+1)a^2\}}{4} e^{2\{A(r)+pF(r)+\frac{a}{2}\phi(r)\}},
\end{aligned} \quad (19)$$

$$\begin{aligned}
& \frac{2n(p+1) + a^2(n+p+1)}{(p+1)^2} \{A''(r) + pF''(r) - \frac{p+1}{a}\phi''(r)\} \\
&= \frac{2(p+1) + a^2}{p+1} \{A''(r) + pF''(r) + nC''(r)\} - 2\{A''(r) + pF''(r) + \frac{a}{2}\phi''(r)\}, \quad (20) \\
& A''(r) - F''(r) = 0. \quad (21)
\end{aligned}$$

Again the first two (18), (19) are Toda equations, and the last two (20), (21) are given by

$$\begin{aligned}
A(r) + pF(r) - \frac{p+1}{a}\phi(r) &= \frac{(p+1)\{2(p+1) + a^2\}}{2n(p+1) + (n+p+1)a^2} \{A(r) + pF(r) + nC(r)\} \\
& - \frac{2(p+1)^2}{2n(p+1) + (n+p+1)a^2} \{A(r) + pF(r) + \frac{a}{2}\phi(r)\} - \kappa r + \mu, \quad (22)
\end{aligned}$$

$$A'(r) - F'(r) + \nu = 0 \Rightarrow A(r) - F(r) = -\nu r + \sigma, \quad (23)$$

where  $\kappa$ ,  $\mu$ ,  $\nu$  and  $\sigma$  are integration constants. Out of eight integration constants, four can be set to some specific values. Then, thanks to the constraint (17) we are left with a  $(4+N_m)$ -parameter family of solutions [17]. More generally, a solution with  $n+p$  nonuniform tensions has  $2+n+p+N_m$  parameters without the assumption of the uniformity.

If we use  $J(r)$  and  $K(r)$ , in which

$$A(r) + pF(r) + \frac{a}{2}\phi(r) = b_1 J(r) + b_2 K(r) + \dots, \quad (24)$$

$$A(r) + pF(r) + nC(r) = b_3 J(r) + b_4 K(r) + \dots, \quad (25)$$

$$\begin{aligned}
b_1 &= \frac{(p+1)\{2n(p+1) + (n+p)a^2\}}{(n+p)\{2n(p+1) + (n+p+1)a^2\}}, \quad b_2 = \frac{n\{2(n-1)(p+1) + (n+p)a^2\}}{(n+p)\{2n(p+1) + (n+p+1)a^2\}}, \\
b_3 &= \frac{(n+p+1)\{2n(p+1) + (n+p)a^2\}}{(n+p)\{2n(p+1) + (n+p+1)a^2\}}, \quad b_4 = -\frac{2n(p+1)}{(n+p)\{2n(p+1) + (n+p+1)a^2\}},
\end{aligned}$$

and  $\dots$  stands for certain appropriate constants, the first two equations (18), (19) can be arranged as follows,

$$J''(r) = e^{2b_1 J(r) + 2b_2 K(r)}, \quad (26)$$

$$K''(r) = e^{2b_3 J(r) + 2b_4 K(r)}. \quad (27)$$

These equations also allow a special solution with  $J(r) = K(r)$  as in the previous section,

$$e^{-J(r)} = e^{-K(r)} = \frac{1}{\gamma} \sinh(\gamma r + \delta). \quad (28)$$

## 4 Discussions

Now let us briefly summarize the main results and possible future directions. In this paper, we have shown that the general equations of motion for static single-scalar multiply-charged branes in AdS space can be written in the form of Toda equations and obtained special brane solutions explicitly. The parameters are mass density, magnetic charges and a coupling constant. In this note, we did not treat a multi-scalar brane case [21, 22], but it would be very interesting to be investigated. An electrically charged brane solution in AdS space and its Born-Infeld generalization would be also interesting since the importance of Born-Infeld terms in the context of extremal black holes and their connection with elementary string states was emphasized [23]. We leave them for future study.

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